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Fuzzy Quantum Spaces and Compatibility

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Using ideas from fuzzy set theory, we generalize the notion of a quantum (probability) space introduced by Suppes to so-called fuzzy quantum spaces and we study the compatibility problem.

1. INTRODUCTION

An important current problem of the mathematical description of quantum mechanics is that of the simultaneous measurement of several observables. In the classical Kolmogorov model (Kolmogorov, 1933) the measurement of (nonquantum) observables is performed within the framework of Boolean σ-algebra models. Today there exists for quantum mechanical observables an axiomatic model of quantum logics (Varadarajan, 1962). Two of the most widely used models of non-Boolean quantum logics are the system of all closed subspaces of a (complex separable) Hilbert space (von Neumann 1932) and a quantum (probability) space introduced by Suppes (1966).

The latter case means the collection of subsets (=quantum mechanical events) of some crisp set X closed with respect to countable disjoint unions and with respect to the complementation, and was introduced in order to describe the position and momentum of a quantum mechanical particle.

Due to one-to-one correspondence between subsets and their characteristic functions, the quantum space of Suppes may be uniquely represented as a system of characteristic functions with values in the closed interval $[0, 1]$. When a quantum mechanical event a, say, is described only vaguely, then by a fuzzy set a (=fuzzy event a) we shall understand a real-valued function a defined on the crisp set X with the values in $[0, 1]$ that describes the fuzziness of the quantum mechanical event a.

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Using the language of fuzzy set theory, we say that

$$\bigcap_i f_i := \inf_i f_i \tag{1.1}$$

$$\bigcup_i f_i := \sup_i f_i \tag{1.2}$$

$$f^\perp := 1 - f \tag{1.3}$$

are called the intersection and the union of the fuzzy sets f_i, and the complement of the fuzzy set f, respectively. Two fuzzy sets f and g are called orthogonal or mutually exclusive and we write $f \perp g$ if $f \leq 1 - g$ (pointwise), and following the ideas of Suppes, we assume that a fuzzy quantum space (for the exact definition see below) is a system of fuzzy sets M, containing the empty set, closed with respect to countable unions of mutually orthogonal fuzzy sets and with respect to the complementation.

The state on M is a mapping $m: M \to [0, 1]$ such that

$$m(f \cup f^\perp) = 1 \qquad \text{for any} \quad f \in M \tag{1.4}$$

$$m\left(\bigcup_i f_i\right) = \sum_i m(f_i) \qquad \text{whenever} \quad f_i \perp f_j \quad \text{for} \quad i \neq j \tag{1.5}$$

The easy consequence of the existence of a state is the excluding of the fuzzy set 1/2 from the quantum space. This is a natural condition because 1/2 denotes "the highest degree of uncertainty."

The main goal of the present paper is to present conditions showing when the ranges of observables in a fuzzy quantum space have classical character, i.e., when their ranges are embeddable into some Boolean σ-algebra. This question is known as the compatibility problem and it has been solved for different classes of quantum logics using different notions of compatibility (e.g., Varadarajan, 1962; Neubrunn, 1970; Gudder, 1979; Brabec, 1979; Brabec and Pták, 1982; Neubrunn and Pulmannová, 1983).

2. FUZZY QUANTUM SPACES

By a fuzzy quantum space we understand a couple (X, M), where X is a nonvoid set and $M \subset [0, 1]^X$ is a system of fuzzy sets such that:

(i) If $0(x) = 0$ for any $x \in X$, then $0 \in M$.
(ii) If $f \in M$, then $1 - f \in M$.
(iii) If $1/2(x) = 1/2$ for any $x \in X$, then $1/2 \notin M$.
(iv) $\bigcup_i f_i \in M$ whenever $f_i \perp f_j$ for $i \neq j$, $f_i \in M$.

A similar structure was studied by Riečan and Dvurečenskij (1986), where instead of condition (iv) it was supposed that M contains the union of any sequence of fuzzy sets; such a structure is known in fuzzy set theory as a fuzzy soft σ-algebra (Piasecki, 1985). Condition (iv) was suggested by Pykacz (1987). The fuzzy quantum logic in the sense of Pykacz (1987) is closed with respect to the sums of orthogonal fuzzy sets and therefore it is different from ours in general. Some ideas of fuzzy sets were studied also by Guz (1985), but with an approach different from ours.

Example 1. Let $X \neq \emptyset$ and let C be a nonempty collection of subsets of X which is closed with respect to the countable unions of disjoint subsets and with respect to the complementation. Then (X, M) is a fuzzy quantum space, where

$$M = \{I_A : A \in C\}$$

(I_A is the indicator of the set A).

Example 2. Let $X_n = \{1, 2, \ldots, 2n\}$, $n \geq 1$, and let M_n be the system of all indicators of all subsets from X_n with an even number of elements. Then (X_n, M_n) is a fuzzy quantum space.

Let (X, M) be a fuzzy quantum space. The set M may be regarded as a partially ordered set in which we define $f \leq g$ if $f(x) \leq g(x)$ for any $x \in X$. Using the complementation $\perp : f \mapsto f^{\perp} = 1 - f$ for any fuzzy set f, we see that $\perp$ satisfies two conditions: (i) $(f^{\perp})^{\perp} = f$ for any $f \in M$; (ii) $f \leq g$ implies $g^{\perp} \leq f^{\perp}$. It is evident that $f \cup f^{\perp} \in M$ for any $f \in M$, and the condition $f \cup f^{\perp} = 1$ for any $f \in M$ is fulfilled iff M consists only of crisp sets. The sup and inf $f \vee g$ and $f \wedge g$, respectively, are defined in the usual way relative to M. Note, however, that $f \vee g$ ($f \wedge g$) need not equal $f \cup g$ ($f \cap g$) even if the former exist; they are equal if the latter are in M. For instance, in Example 2, put $X_2 = \{1, 2, 3, 4\}$. Then M_2 is a lattice, but if $A = \{1, 2\}$ and $B = \{1, 3\}$, then $I_A \cap I_B = I_{\{1\}} \notin M$ and $I_A \wedge I_B = 0$. On the other hand, M_n, for any $n \geq 3$, is not a lattice.

A nonempty set $A \subset M$ is said to be a Boolean algebra (σ-algebra) of a fuzzy quantum space (X, M) if:

1. There are minimal and maximal elements 0_A and 1_A from A such that, for any $f \in A$, $0_A \leq f \leq 1_A$ and $f \cup f^{\perp} = 1_A$.
2. With respect to $\cap$, $\cup$, $\perp$, 0_A, and 1_A, A is a Boolean algebra (σ-algebra) [in the sense of Sikorski (1964)].

It is clear that $0_A \neq 1_A$.

Let $\mathscr{A}$ be a Boolean (σ-) algebra. We say that a mapping $x : \mathscr{A} \to M$ is an $\mathscr{A}$-observable ($\mathscr{A}$-σ-observable) of (X, M) if:

(i) $x(A') = x(A)^{\perp}$, $A \in \mathscr{A}$, where A' denotes the complement of A in a Boolean algebra $\mathscr{A}$.

(ii) $x(A) \perp x(B)$ whenever $A \wedge B = 0$, $A, B \in \mathscr{A}$.

(iii) $x(A \vee B) = x(A) \cup x(B)$ if $A \wedge B = 0$; $x(\bigvee_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} x(A_i)$ whenever $A_i \wedge A_j = 0$ for $i \neq j$.

Of great importance for quantum mechanics is the case when $\mathscr{A}$ is a Borel σ-algebra of some separable Banach space Y, in particular, when $Y = R_1$ (the set of all reals). In this case, for brevity a $B(R_1)$- (σ-) observable of (X, M) will be called a (σ-) observable of (X, M).

If x is an $\mathscr{A}$- (σ-) observable, then the range of x, that is, the set $\mathscr{R}(x) = \{x(A): A \in \mathscr{A}\}$, is a Boolean algebra ($\sigma$-algebra) of a fuzzy quantum space (X, M) with the minimal and maximal elements $x(0)$ and $x(1)$, respectively.

Let f be a fuzzy set of (X, M). We define a question observable x_f as a mapping from $B(R_1)$ into M such that

$$x_f(E) = \begin{cases} f \cap f^{\perp} & \text{if } 0, 1 \notin E \\ f^{\perp} & \text{if } 0 \in E, 1 \notin E \\ f & \text{if } 0 \notin E, 1 \in E \\ f \cup f^{\perp} & \text{if } 0, 1 \in E \end{cases}$$

for any $E \in B(R_1)$. It is evident that x_f is a σ-observable and it plays the role of the indicator of the fuzzy set $f \in M$.

In accordance with the theory of quantum logics, we say that two elements $f, g \in M$ are (i) compatible and write $f \leftrightarrow g$ if $f \cap g, f \cap g^{\perp}, f^{\perp} \cap g \in M$, and $f = f \cap g \cup f \cap g^{\perp}$, $g = f \cap g \cup f^{\perp} \cap g$; and (ii) strongly compatible and write $f \leftrightarrow^s g$ if $f \leftrightarrow g \leftrightarrow f^{\perp} \leftrightarrow g^{\perp} \leftrightarrow f$ (it suffices that only $f \leftrightarrow g \leftrightarrow f^{\perp} \leftrightarrow g^{\perp}$).

It is evident that if $f \leftrightarrow g$, then $f \cup g \in M$, and $f \leftrightarrow^s f$, $f \cap f^{\perp} \leftrightarrow^s f \leftrightarrow^s f \cup f^{\perp} \leftrightarrow^s f \cup f^{\perp}$ for any $f \in M$.

We note that if $f \leftrightarrow g$, then it does not imply $f \leftrightarrow^s g$ in general. Indeed, let M consist of constant functions from $[0, 0.49] \cup [0.51, 1]$, and let f and g be two functions such that $0 < f < g < 1/2$. Then $f \leftrightarrow g$ and $f \not\leftrightarrow^s g$. Analogously, it is not true that if $f \leftrightarrow g$ and $f \leftrightarrow g^{\perp}$, then $f \leftrightarrow^s g$. In the same example we shall show that if there are three mutually orthogonal fuzzy sets f_1, g_1, and h such that $f = f_1 \cup h$, $g = g_1 \cup h$, then it is not true that $f_1 = f \cap g^{\perp}$, $g_1 = g \cap f^{\perp}$, and $h = f \cap g$ [for comparison with quantum logics see Varadarajan (1962)].

Lemma 2.1. For a fuzzy quantum space (X, M) the following statements are equivalent:

(i) $f \leftrightarrow^s g$.
(ii) $f \cup f^{\perp} = g \cup g^{\perp}$ and $f \cap g, f \cap g^{\perp}, f^{\perp} \cap g, f^{\perp} \cap g^{\perp} \in M$.

(iii) There is an observable x of (X, M) such that $x(E)=f$, $x(F)=g$ for some $E, F \in B(R_1)$.
(iv) There is a Boolean algebra of (X, M) containing f and g.

Proof. Let (i) hold. The distributivity of the union and the intersection for fuzzy sets implies

$$f = f \cap g \cup f \cap g^{\perp} = f \cap (g \cup g^{\perp})$$

which gives $f \leq g \cup g^{\perp}$; analogously, $f^{\perp} \leq g \cup g^{\perp}$. Hence, $f \cup f^{\perp} \leq g \cup g^{\perp}$. The symmetry entails $f \cup f^{\perp} = g \cup g^{\perp}$.

Now let (ii) hold. Then, for example,

$$f \cap g \cup f \cap g^{\perp} = f \cap (g \cup g^{\perp}) = f \cap (f \cup f^{\perp}) = f$$

Similarly we proceed for other equations.

Suppose (iii). Put $x_1 = f \cap g$, $x_2 = f \cap g^{\perp}$, $x_3 = f^{\perp} \cap g$, and $x_4 = f^{\perp} \cap g^{\perp}$ and define a mapping $x: B(R_1) \to M$ via

$$x(A) = \begin{cases} f \cap f^{\perp} & \text{if} \quad 1, 2, 3, 4 \notin A \\ \bigcup \{x_i : \text{if } i \in A \text{ , } i = 1, 2, 3, 4\} & \text{otherwise} \end{cases}$$

for any $A \in B(R_1)$. Straightforward calculation shows that x is an observable of (X, M). If we put $E = \{1, 2\}$ and $F = \{1, 3\}$, then we get (iii).

The statement (iii) evidently gives (iv), and (iv) implies (i). ∎

We note that in part (ii) of Lemma 2.1, it suffices to suppose only the existence of three arbitrary fuzzy intersections. Indeed, let, for instance, the first three intersections be in M. Then

$$f^{\perp} \cap g^{\perp} = (f \cup g)^{\perp} = (f \cap g \cup f \cap g^{\perp} \cup f^{\perp} \cap g)^{\perp} \in M$$

3. COMMENSURABILITY

We say that two nonempty subsets A and B of M are compatible (strongly compatible) and write $A \leftrightarrow B$ ($A \leftrightarrow^s B$) if $a \leftrightarrow b$ ($a \leftrightarrow^s b$) for all $a \in A$, $b \in B$. If A and B are Boolean algebras of a fuzzy quantum space (X, M), then $A \leftrightarrow B$ iff $A \leftrightarrow^s B$. We say that a system of nonempty subsets of M, $\{A_t : t \in T\}$, is (σ-) commensurable if there is a Boolean (σ-) algebra of (X, M) containing all A_t.

The main problem of the present section is to give the necessary and sufficient conditions (=compatibility theorem) in order for $\{A_t : t \in T\}$ be (σ-) commensurable.

If A and B are two Boolean algebras of (X, M) with $A \cap B \neq \emptyset$, then $0_A = 0_B$ and $1_A = 1_B$. Indeed, for any $f \in A \cap B$, we have $1_A = f \cup f^{\perp} = 1_B$.

Hence, if a nonvoid subset C is (σ-) commensurable, then there is a minimal Boolean (σ-) algebra of (X, M) containing C.

A nonvoid subset A of M is said to be f-compatible ("f" for finiteness) if for all $f_1, \dots, f_{n+1} \in A$ we have (i) $b_1 := f_1 \cap \cdots \cap f_n \cap f_{n+1} \in M$, $b_2 := f_1 \cap \cdots \cap f_n \cap f_{n+1}^{\perp} \in M$; (ii) $b_1 \cup b_2 = f_1 \cap \cdots \cap f_n$. The subset A is strongly f-compatible if $A \cup A^{\perp}$ is f-compatible, where $A^{\perp} = \{f^{\perp} : f \in A\}$.

Lemma 3.0. Let $A = \{f_1, \dots, f_n\}$ be any finite subset of M, where (X, M) is a fuzzy quantum space from Example 1. The following statements are equivalent:

(i) A is strongly f-compatible.
(ii) A is f-compatible.
(iii) $f_{i_1} \cap \cdots \cap f_{i_j} \in M$ for any subset $\{i_1, \dots, i_j\}$ of $\{1, \dots, n\}$.

Proof. The implications (i) → (ii) → (iii) are evident. Suppose now (iii) holds. Since M is in our case an orthomodular σ-orthoposet, we conclude that if $f, g \in M$ and $f \le g$, then $g \cap f^{\perp} = (g^{\perp} \cup f)^{\perp} \in M$. Hence, for all i, j,

$$f_i \cap (f_i \cap f_j)^{\perp} = f_i \cap f_j^{\perp} \in M$$

$$f_i^{\perp} \cap f_j^{\perp} = (f_i \cup f_j)^{\perp} = (f_i \cap f_j^{\perp} \cup f_i \cap f_j \cup f_j \cap f_i^{\perp})^{\perp} \in M$$

Analogously, we may prove that condition (iii) implies $f_i^{k_1} \cap \cdots \cap f_i^{k_j} \in M$ for any $k_1, \dots, k_n \in \{0, 1\}$, where $f^0 = f^{\perp}$, $f^1 = f$. ■

Proposition 3.1. Let $\bigcup_i f_i \in M$ and $\bigcup_i (g \cap f_i) \in M$. Then

$$g \cap \bigcup_i f_i = \bigcup_i (g \cap f_i) \tag{3.1}$$

In particular, if f_i are mutually orthogonal and $g \cap f_1 \in M$ for any i, then (3.1) holds.

Proof. The distributivity of the intersections and the unions of fuzzy sets gives the assertion of the proposition. ■

Proposition 3.2. (i) $a \leftrightarrow b$ iff $\{a, b\}$ is f-compatible.

(ii) If A is f-compatible, then any nonvoid subset of A is f-compatible.

(iii) The f-compatiblity of $\{f_1, \dots, f_n\}$ implies $\bigcap_{i=1}^n f_i \in M$; the strong f-compatibility of $\{f_1, \dots, f_n\}$ gives $\bigcup_{i=1}^n f_i \in M$.

(iv) If $\{g, f_i, \dots, f_n\}$ is strongly f-compatible, then $g \leftrightarrow^s \bigcup_{i=1}^n f_i$ and $g \leftrightarrow^s \bigcap_{i=1}^n f_i$.

Proof. The first two statements are evident. From the definition we have easily $\bigcap_{i=1}^n f_i \in M$. Suppose now $\{f_1, \dots, f_n\}$ is strongly f-compatible.

Then $\{f_1^{\perp}, \ldots, f_n^{\perp}\}$ is f-compatible and therefore

$$\left(\bigcup_{i=1}^{n} f_i\right)^{\perp} = \bigcap_{i=1}^{n} f_i^{\perp} \in M$$

(iv) Straightforward calculation shows that

$$f_i = \bigcup_{\substack{j_1 \cdots j_n = 0 \\ j_i = 1}}^{1} f_1^{j_1} \cap \cdots \cap f_n^{j_n} \tag{3.2}$$

where $f^0 = f^{\perp}$, $f^1 = f$. Then

$$\bigcup_{i=1}^{n} f_i = \bigcup_{D} f_1^{j_1} \cap \cdots \cap f_n^{j_n} \tag{3.3}$$

where D denotes the summation over $(i_i, \ldots, i_n) \in \{0, 1\}^n - \{(0, \ldots, 0)\}$. Applying Proposition 3.1 to g and $f_1^{j_1} \cap \cdots \cap f_n^{j_n}$, we see that

$$\begin{aligned}
g \cap \left(\bigcup_{i=1}^{n} f_i\right) \cup g^{\perp} \cap \left(\bigcup_{i=1}^{n} f_i\right) \\
&= \bigcup_{i=1}^{n} (g \cap f_i) \cup \bigcup_{i=1}^{n} (g^{\perp} \cap f_i) \\
&= \bigcup_{i=1}^{n} (g \cap f_i \cup g^{\perp} \cap f_i) = \bigcup_{i=1}^{n} f_i
\end{aligned}$$

$$\begin{aligned}
g \cap \left(\bigcup_{i=1}^{n} f_i\right) \cup g \cap \left(\bigcup_{i=1}^{n} f_i\right)^{\perp} \\
&= \bigcup_{i=1}^{n} (g \cap f_i) \cup g \cap \bigcap_{i=1}^{n} f_i^{\perp} \\
&= \bigcup_{D} g \cap f_1^{j_1} \cap \cdots \cap f_n^{j_n} \cup g \cap f_1^{\perp} \cap \cdots \cap f_n^{\perp} \\
&= \bigcup_{j_1 \cdots j_n = 0}^{1} g \cap f_i^{j_1} \cap \cdots \cap f_n^{j_n} \\
&= \bigcup_{j_1 \cdots j_{n-1} = 0}^{1} g \cap f_1^{j_1} \cap \cdots \cap f_{n-1}^{j_{n-1}} = \cdots = g \cap f_1 \cup g \cap f_1^{\perp} = g
\end{aligned}$$

Hence, $g \leftrightarrow \bigcup_{i=1}^{n} f_i$, so that $g^{\perp} \leftrightarrow \bigcup_{i=1}^{n} f_i$. Analogously as above, we show that

$$g \cap \bigcap_{i=1}^{n} f_i \cup g \cap \left(\bigcup_{i=1}^{n} f_i^{\perp}\right) = g$$

Since

$$g \cap \bigcap_{i=1}^{n} f_i \cup g^{\perp} \bigcap_{i=1}^{n} f_i = \bigcap_{i=1}^{n} f_i$$

we conclude $g \leftrightarrow \bigcap_{i=1}^{n} f_i$. Applying this result to $\{g, f_1^{\perp}, \ldots, f_n^{\perp}\}$, we see that $g \leftrightarrow \bigcup_{i=1}^{n} f_i^{\perp} = (\bigcup_{i=1}^{n} f_i)^{\perp}$, which gives finally $g \leftrightarrow^s \bigcup_{i=1}^{n} f_i$ and $g \leftrightarrow^s \bigcap_{i=1}^{n} f_i$. ■

Theorem 3.3. (Compatibility theorem). Let A be a nonempty set of a fuzzy quantum space (X, M). The following statements are equivalent:

(i) A is strongly f-compatible.
(ii) A is commensurable.
(iii) A is σ-commensurable.

The proof of the theorem depends on the following results.

Lemma 3.4. Any two compatible Boolean algebras of (X, M) are commensurable.

Proof. Let A and B be two compatible Boolean algebras of (X, M). Then

$$1_A = 1_A \cap 1_B \cup 1_A \cap 0_B = 1_A \cap (1_B \cup 0_B) = 1_A \cap 1_B$$

Analogously, $1_B = 1_A \cap 1_B$, so that $1_A = 1_B$.

Define $\mathscr{D} = \{f \wedge g : f \in A, g \in B\}$. Then

$$\begin{aligned} f \cap g \cup (f \cap g)^{\perp} &= f \cap g \cup f^{\perp} \cup g^{\perp} \\ &= f \cap g \cap f^{\perp} \cap g \cup f^{\perp} \cap g^{\perp} \\ &\quad \cup f \cap g^{\perp} \cup f^{\perp} \cap g^{\perp} \\ &= g \cup g^{\perp} = 1_A. \end{aligned}$$

Let $u \in A$ and $v \in B$ be arbitrary elements. Then

$$f \cap g \cap u \cap v = (f \cap u) \cap (g \cap v) \in M$$

$$\begin{aligned} f \cap g \cap (u \cap v)^{\perp} &= f \cap g \cap u^{\perp} \cup f \cap g \cap v^{\perp} \\ &= f \cap g \cap u^{\perp} \cap v \cup f \cap g \cap u^{\perp} \cap v^{\perp} \cup f \cap g \\ &\quad \cap u \cap v^{\perp} \cup f \cap g \cap u^{\perp} \cap v^{\perp} \in M \end{aligned}$$

Analogously, $(f \cap g)^{\perp} \cap u \cap v \in M$. In view of part (ii) of Lemma 2.1, $f \cap g \leftrightarrow^s u \cap v$. Denote by $\mathscr{R} = \{\bigcup_{i=1}^{n} f_i : f_i \in \mathscr{D}, f_i \perp f_j \text{ for } i \neq j, n \geq 1\}$. We claim to show that $\mathscr{R}$ is a minimal Boolean algebra of (X, M) containing A and B.

Let $f = \bigcup_{i=1}^{n} f_i \in \mathscr{R}$, where $f_i \perp f_j$, $i \neq j$, and let $g \in \mathscr{D}$.

From Proposition 3.1, $f \cap g = \bigcup_{i=1}^{n} (f_i \cap g) \in \mathscr{R}$.

Suppose now that $f=\bigcup_{i=1}^n f_i\in\mathcal{R}$ and $g=\bigcup_{j=1}^m g_j\in\mathcal{R}$, where $f_i\perp f_k$, $g_j\perp g_s, f_i, g_j\in\mathcal{D}$. Then $(\bigcup_{i=1}^n f_i)\cap g_j\in\mathcal{R}$, and it is orthogonal with $(\bigcup_{i=1}^n f_i)\cap g_s$ for $j\neq s$. Then, due to Proposition 3.1,

$$f\cap g=\bigcup_{j=1}^m\left(\bigcup_{i=1}^n f_i\right)\cap g_j=\bigcup_{i=1}^n\bigcup_{j=1}^m f_i\cap g_j\in\mathcal{R}$$

Now we show that if $h\in\mathcal{R}$, then $h^\perp\in\mathcal{R}$. First, let $h=f\cap g$, where $f\in A$, $g\in B$. Then

$$\begin{aligned}h^\perp&=f^\perp\cup g^\perp=(f^\perp\cap g\cup f^\perp\cap g^\perp)\cup(f\cap g^\perp\cup f^\perp\cap g^\perp)\\&=f^\perp\cap g\cup f\cap g^\perp\cup f^\perp\cap g^\perp\in\mathcal{R}\end{aligned}$$

If $h=\bigcup_{i=1}^n f_i\in\mathcal{R}$, where $f_i\in\mathcal{D}$, $f_i\perp f_j$ for $i\neq j$, then $h^\perp=\bigcap_{i=1}^n f_i^\perp\in\mathcal{R}$.

In view of the above and part (ii) of Lemma 2.1, to show that $\mathcal{R}$ is a Boolean algebra of (X, M), it is sufficient to prove $f\cup f^\perp=1_A$ for any $f\in\mathcal{R}$. Calculate

$$f\cup f^\perp=\bigcap_{j=1}^n(f\cup f_j^\perp)=\bigcap_{j=1}^n\left(\bigcup_{i=1}^n f_i\cup f_j^\perp\right)$$

Since $f_i\cup f_j^\perp\le 1_A$ for all i and j and $f_i\cup f_i^\perp=1_A$, we see that $f\cup f^\perp=1_A$. ∎

Lemma 3.5. Let A, B, and C be three Boolean algebras of (X, M) such that $A\cup B\cup C$ is f-compatible. Denote by $A\vee B$ the minimal Boolean algebra of (X, M) containing A and B. Then $C\leftrightarrow A\vee B$.

Proof. Let $f\in A$, $g\in B$, $h\in C$. According to part (iv) of Proposition 3.2, $h\leftrightarrow^s f\wedge g$. Therefore, if $u=\bigcup_{i=1}^n f_i\cap g_i$, where $f_i\cap g_i\perp f_j\cap g_j$ for $i\neq j, f_i\in A$, $g_i\in B$, then

$$h\cap\bigcup_{i=1}^n(f_i\cap g_i)=\bigcup_{i=1}^n(h\cap f_i\cap g_i)\in M$$

so that $h\cap u\in M$ for any $u\in\mathcal{R}$ and this yields $h\leftrightarrow^s u$ if we use part (ii) of Lemma 2.1. ∎

Lemma 3.6. Let $A_1,\ldots,A_n$ be Boolean algebras of (X, M). Then $\bigcup_{i=1}^n A_i$ is f-compatible iff $A_1,\ldots,A_n$ are pairwise compatible, and for any $a_i\in A_i$, $i=1,\ldots,n$, $a_1\cap\cdots\cap a_n\in M$.

Proof. It is evident that $a_1\cap\cdots\cap a_i\in M$. Hence,

$$\begin{aligned}&a_1\cap\cdots\cap a_i\cap a_{i+1}\cup a_1\cap\cdots\cap a_i\cap a_{i+1}^\perp\\&\quad=a_1\cap\cdots\cap a_i\cap(a_{i+1}\cup a_{i+1}^\perp)=a_1\cap\cdots\cap a_i\end{aligned}$$

The general case is based on the same arguments. ∎

Lemma 3.7. Let $A_1,\ldots,A_n$ be Boolean algebras of (X, M) such that $\bigcup_{i=1} A_i$ if f-compatible. Then $A_1,\ldots,A_n$ are commensurable.

Proof. The statement of the lemma follows from the observation that the minimal Boolean algebra of (X, M) containing $A_1, \ldots, A_n$ consists of the elements of the form $\bigcup_{i=1}^{n} (f_1^i \cap \cdots \cap f_n^i)$, where $f_k^i \in A_k$ for $k = 1, \ldots, n$ and $f_1^i \cap \cdots \cap f_n^i \perp f_1^j \cap \cdots \cap f_n^j$ for $i \neq j$. To prove that, we use the same arguments as in the proof of Lemma 3.4. ■

The statement of Lemma 3.7 is incorrect if we assume only the mutual compatibility of $A_1, \ldots, A_n$. Indeed, in Example 2 choose x_4 and M_4 and let $f = I_{\{1,2,3,4\}}$, $g = I_{\{1,2,5,6\}}$, and $h = I_{\{1,3,5,7\}}$. Then $A = \{0, f, f^\perp, 1\}$, $B = \{0, g, g^\perp, 1\}$, and $C = \{0, h, h^\perp, 1\}$ are pairwise compatible Boolean algebras of (X_4, M_4). But $f \cap g \cap h = I_{\{1\}} \notin M_4$. So A, B, and C are not commensurable.

In addition, we note that for any integer $n \geq 2$, there is a fuzzy quantum space (X^n, M^n) and a subset A_n of M^n such that A_n contains exactly n points and (i) A_n is f-compatible and (ii) any subset and A_n consisting of at least $n-1$ fuzzy sets if f-compatible.

This assertion follows from Theorem 3 of Brabec and Pták (1982) and Lemma 2.2. This result establishes that our definition of a (strong) f-compatibility cannot be restated in a more economical form.

Lemma 3.8. Let $\{A_t : t \in T\}$ be a system of Boolean algebras of a fuzzy quantum space (X, M) such that $\bigcup_{t \in T} A_t$ is f-compatible. Then the system $\{A_t : t \in T\}$ is commensurable.

Proof. Let T_0 be any finite, nonempty subset of T. In view of Lemma 3.7, there is a minimal Boolean algebra $A(T_0)$ of (X, M) containing all A_t for $t \in T_0$. Write $A = \bigcup \{A(T_0)\colon T_0$ is a finite subset of $T\}$. Simple verification shows that A is a Boolean algebra of (X, M) including all A_t, $t \in T$. ■

Lemma 3.9. Any Boolean algebra of (X, M) is contained in a maximal one. A maximal Boolean algebra of (X, M) is necessarily a Boolean σ-algebra. Commensurability and σ-commensurability are equivalent notions.

Proof. The Zorn lemma easily implies the first statement. In order to show the second suppose A is a maximal Boolean algebra of (X, M). Let $\{f_n\}_{n=1}^{\infty}$ be an arbitrary sequence of mutually orthogonal elements from A. Put $f = \bigcup_{n=1}^{\infty} f_n \in M$. First we show that $f \cup f^\perp = 1_A$, where 1_A is the maximal element in A. Using the distributivity of the intersections and unions of fuzzy sets, we obtain

$$f \cup f^\perp = \bigcup_{n=1}^{\infty} f_n \cup \bigcap_{m=1}^{\infty} f_m^\perp = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} f_n \cup f_m^\perp = 1_A$$

since for all n and m, $f_n \cup f_m^\perp \leq 1_A$ and $f_n \cup f_n^\perp = 1_A$.

Now let a be an arbitrary fuzzy set from A; we claim to show $a \leftrightarrow^s f$. According to the criterion 2.1, we show $a \cap f$, $a \cap f^\perp \in M$. Calculate

$$a \cap f = a \cap \bigcup_{n=1}^{\infty} f_n = \bigcup_{n=1}^{\infty} (a \cap f_n)$$

Due to the mutual orthogonality of $\{a \cap f_n\}$, we conclude $a \cap f \in M$.

Define $g_0 = a^\perp$, $g_n = f_n \cap (\bigcup_{i=1}^{n-1} g_i) \in A$ for $n \geq 1$. Then $\{g_n\}_{n=0}^{\infty}$ are mutually orthogonal elements of A and $\bigcup_{i=0}^{n} g_i = \bigcup_{i=1}^{n} (a^\perp \cup f_i)$ for any $n \gtrsim 1$. Hence,

$$\begin{aligned} a \cap f^\perp &= \bigcap_{n=1}^{\infty} (a \cap f_n^\perp) \\ &= \left[\bigcup_{n=1}^{\infty} (a^\perp \cup f_n)\right]^\perp \\ &= \left[\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n} (a^\perp \cup f_i)\right]^\perp \\ &= \left(\bigcup_{n=0}^{\infty} \bigcup_{i=0}^{n} g_i\right)^\perp \\ &= \left(\bigcup_{n=0}^{\infty} g_n\right)^\perp \in M \end{aligned}$$

The equivalence of commensurability and σ-commensurability is now evident. ■

Proof of Theorem 3.3. For any $f \in A$, define a Boolean algebra A_f of (X, M) via $A_f = \{f \cap f^\perp, f^\perp, f, f \cup f^\perp\}$. The conditions of the theorem yield $f \in A$, A_f is f-compatible. Appealing to Lemmas 3.8 and 3.9, the proof is finished. ■

4. COMPATIBLE OBSERVABLES

Here we apply the results of the last section to the problem of the existence of joint σ-observables in a general form. Dvurečenskij and Riečan (1987) solved this problem only for $B(R_1)$-σ-observables. On the other hand, it is known (Sikorski, 1964) that if the assumption on $B(R_1)$ is omitted, then the joint σ-observable may fail even for two compatible observables. Here we show that for f-compatible observables in any fuzzy quantum space it always exists.

We say that a system $\{\mathscr{A}_t: t\in T\}$ of Boolean sub- (σ-) algebras of a Boolean- (σ-) algebra $\mathscr{A}$ is independent (σ-independent) if, for any finite (countable) subset $\alpha\subset T$

$$\bigwedge_{t\in\alpha} A_t\neq 0$$

for any $0\neq A_t\in\mathscr{A}_t$ and any $t\in\alpha$.

For example, let $(\Omega_t,\mathscr{S}_t)$, $t\in T$, be a measure space, that is, $\mathscr{S}_t$ is a (σ-) algebra of subsets of a set $\Omega_t\neq\varnothing$. Denote by $\Omega=\prod_{t\in T}\Omega_t$ the Cartesian product of all sets Ω_t, i.e., the set of all $\omega=(\omega_t: t\in T)$, where $\omega_t\in\Omega_t$ for any $t\in T$. Let Π_t be the tth projection function from Ω onto Ω_t, that is, $\Pi_t\omega=\omega_t$, $\omega\in\Omega$. Let $\mathscr{S}_t^*=\{\Pi_t^{-1}(A): A\in\mathscr{S}_t\}$, $t\in T$. Then $\mathscr{S}_t$ is isomorphic $\mathscr{S}_t^*$. The minimal (σ-) algebra of Ω generated by all $\mathscr{S}_t^*$ is denoted by $\mathscr{S}=\prod_{t\in T}\mathscr{S}_t$, and $\{\mathscr{S}_t^*: t\in T\}$ is a system of (σ-) independent sub-(σ-) algebras of $\mathscr{S}$.

Let $\{\mathscr{A}_t: t\in T\}$ be a system of (σ-) independent Boolean sub-(σ-) algebras of a Boolean (σ-) algebra $\mathscr{A}$. Denote by $\mathscr{D}=\mathscr{D}(T)$ the system of all Boolean rectangles $\bigwedge_{t\in\alpha}A_t$ defined for any $A_t\in\mathscr{A}_t$, $t\in\alpha$, and each finite nonempty subset $\alpha\subset T$. As in the Cartesian product of (σ-) algebras of subsets of Ω_t, one may verify that the minimal subalgebra $\mathscr{R}=\mathscr{R}(T)$ of $\mathscr{A}$ generated by all $\mathscr{A}_t$, $t\in T$, consists of all finite joins of orthogonal elements from $\mathscr{D}$. The minimal sub-σ-algebra $\mathscr{A}(T)$ of $\mathscr{A}$ generated by all sub-σ-algebras $\mathscr{A}_t$, $t\in T$, is denoted by $\mathscr{A}(T)=\prod_{t\in T}\mathscr{A}_t$.

Finally, we make the following simple observations. Any two Boolean rectangles $\bigwedge_{t\in\alpha}A_t$ and $\bigwedge_{s\in\beta}B_s$ can be assumed on the same finite index subset $\alpha\cup\beta$. Indeed, if we put $A_t^*=A_t$ if $t\in\alpha$, $A_t^*=1$ if $t\in\beta-\alpha$, and $B_t^*=B_t$ if $t\in\beta-\alpha$, $B_t^*=1$ if $t\in\alpha$, then $\bigwedge_{t\in\alpha}A_t=\bigwedge\{A_t^*: t\in\alpha\cup\beta\}$ and $\bigwedge_{s\in\beta}B_s=\bigwedge\{B_t^*: t\in\alpha\cup\beta\}$. Therefore, (i) $\bigwedge_{t\in\alpha}A_t=0$, $A_t\in\mathscr{A}_t$, $t\in\alpha$, iff at least one $A_t=0$; (ii) $0\neq\bigwedge_{t\in\alpha}A_t\leq\bigwedge_{t\in\alpha}B_t$ iff $A_t\leq B_t$ for any $t\in\alpha$; (iii) $0\neq\bigwedge_{t\in\alpha}A_t=\bigwedge_{t\in\alpha}B_t$ iff $A_t=B_t$ for any $t\in\alpha$.

Let $\{\mathscr{A}_t: t\in T\}$ be a system of σ-independent Boolean sub-σ-algebras of Boolean σ-algebra $\mathscr{A}$. We say that a system $\{x_t: t\in T\}$, where x_t is an $\mathscr{A}_t$-σ-observable of a fuzzy quantum space (X, M), has a joint σ-observable if there is an $\mathscr{A}(T)$-σ-observable x of (X, M) such that

$$x\left(\bigwedge_{t\in\alpha}A_t\right)=\bigcap_{t\in\alpha}x_t(A_t) \tag{4.1}$$

for any $A_t\in\mathscr{A}_t$ and any finite nonempty subset $\alpha\subset T$, supposing that the fuzzy set intersection on the right-hand side of (4.1) exists in M. It is clear that if the joint σ-observable exists for $\{x_t: t\in T\}$, then it is unique.

Let $\mathscr{A}$ and $\mathscr{B}$ be two Boolean algebras. We recall that an $\mathscr{A}$-observable x and a $\mathscr{B}$-observable y are compatible if $x(A)\leftrightarrow y(B)$ for any $A\in\mathscr{A}$,

$B \in \mathscr{B}$. Analogously we say that $\{x_t : t \in T\}$ is a system of f-compatible observables if $\bigcup_{t \in T} \mathscr{R}(x_t)$ is an f-compatible set in M.

Theorem 4.1. Let $\{\mathscr{A}_t : t \in T\}$ be a system of independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathscr{A}$. For any $t \in T$, let x_t be an $\mathscr{A}_t$-σ-observable of a fuzzy quantum system (X, M). Then $\{x_t : t \in T\}$ has a joint σ-observable iff $\{x_t : t \in T\}$ are f-compatible.

Proof. Let x be a joint σ-observable of $\{x_t : t \in T\}$, and denote by $\mathscr{R}(x)$ the range of x. Then, due to (4.1), $\mathscr{R}(x_t) \subseteq \mathscr{R}(x)$, so that $\{x_t : t \in T\}$ are f-compatible.

Conversely, let $\{x_t : t \in T\}$ be f-compatible. In view of Theorem 3.3, $\{\mathscr{R}(x_t) : t \in T\}$ are commensurable, so that there is a minimal Boolean σ-algebra A of (X, M) containing all ranges of observables. Define a mapping $x : \mathscr{D}(T) \to A \subseteq M$ via (4.1). The mapping x is well-defined.

We note that any Boolean σ-algebra of (X, M) (and therefore also A) is σ-distributive. This means that, for any two-indexed sequence $\{f_{n,m}\}_{n,m=1}^{\infty}$ of elements from A, we have

$$\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} f_{n,m} = \bigcup_{\varphi \in N^N} \bigcap_{n=1}^{\infty} f_{n,\varphi(n)} \tag{4.2}$$

where $N = \{1, 2, \ldots\}$. According to Sikorski (1964, Theorem 19.1), to verify (4.2) it is necessary to show that if $\{f_n\}_{n=1}^{\infty} \subset A$ is given, then, for any $0 \neq f \in A$, there is a sequence $\{a(n)\}_{n=1}^{\infty}{}_{n=1}^{\infty}$ of elements from $\{0, 1\}$ such that

$$f \cap \bigcap_{n=1}^{\infty} f_n^{d(n)} \neq 0 \tag{4.3}$$

(here $f_n^0 = f_n^{\perp}$, $f_n^1 = f_n$). Using the pointwise properties of fuzzy sets, it is evident that (4.3) holds. According to Sikorski (1964, Theorem 37.1), the mapping x may be uniquely extended to an $\mathscr{A}(T)$-σ-observable of (X, M). ■

In conclusion, we note that for a sequence of σ-observables $\{x_n\}$ we may build up the calculus for f-compatible observables in the manner of Varadarajan (1962): there is a sequence of Borel measurable real-valued functions $\{f_n\}$ and a σ-observable x such that $x_n = f_n \circ x : E \mapsto x(f_n^{-1}(E))$, $E \in B(R_1)$ for any n.

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